

Information Complexity versus Corruption and Applications to Orthogonality and Gap-Hamming *

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Abstract

Three decades of research in communication complexity have led to the invention of a number of techniques to lower bound randomized communication complexity. The majority of these techniques involve properties of large submatrices (rectangles) of the truth-table matrix defining a communication problem. The only technique that does not quite fit is information complexity, which has been investigated over the last decade. Here, we connect information complexity to one of the most powerful “rectangular” techniques: the recently-introduced smooth corruption (or “smooth rectangle”) bound. We show that the former subsumes the latter under rectangular input distributions. We conjecture that this subsumption holds more generally, under arbitrary distributions, which would resolve the long-standing direct sum question for randomized communication.

As an application, we obtain an optimal $\Omega(n)$ lower bound on the information complexity—under the *uniform distribution*—of the so-called orthogonality problem (ORT), which is in turn closely related to the much-studied Gap-Hamming-Distance (GHD). The proof of this bound is along the lines of recent communication lower bounds for GHD, but we encounter a surprising amount of additional technical detail.

1 Introduction

The basic, and most widely-studied, notion of communication complexity deals with problems in which two players—Alice and Bob—engage in a communication protocol designed to “solve a problem” whose input is split between them. We shall focus exclusively on this model here, and we shall be primarily concerned with the problem of computing a Boolean function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{-1, 1\}$. As is often the case, we are most interested in lower bounds.

1.1 Lower Bound Techniques and the Odd Man Out

The preeminent textbook in the field remains that of Kushilevitz and Nisan [KN97], which covers the basics as well as several advanced topics and applications. Scanning that textbook, one finds a number of lower bounding techniques, i.e., techniques for proving lower bounds on $D(f)$ and $R(f)$, the deterministic and randomized (respectively) communication complexities of f . Some of the more important techniques are the fooling set technique, log rank, discrepancy and corruption.¹ Research postdating the publication of the

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¹Though the corruption technique is discussed in Kushilevitz and Nisan, the term “corruption” is due to Beame et al. [BPSW06]. The technique has also been called “one-sided discrepancy” and “rectangle method” [Kla03] by other authors.

book has produced a number of other such techniques, including the factorization norms method [LS09], the pattern matrix method [She08], the partition bound and the smooth corruption² bound [JK10]. Notably, all of these techniques ultimately boil down to a fundamental fact called the *rectangle property*. One way of stating it is that each *fiber* of a deterministic protocol, defined as a maximal set of inputs $(x, y) \in \mathcal{X} \times \mathcal{Y}$ that result in the same communication transcript, is a combinatorial rectangle in $\mathcal{X} \times \mathcal{Y}$. The aforementioned lower bound techniques ultimately invoke the rectangle property on a protocol that computes f ; for randomized lower bounds, (the easy direction of) Yao’s minimax lemma also comes into play.

One recent technique is an odd man out: namely, *information complexity*, which was formally introduced by Chakrabarti et al. [CSWY01], generalized in subsequent work [BJKS04, JKS03, BBCR10], though its ideas appear in the earlier work of Abloyev [Abl96] (see also Saks and Sun [SS02]). Here, one defines an *information cost* measure for a protocol that captures the “amount of information revealed” during its execution, and then considers the resulting complexity measure $IC(f)$, for a function f . A precise definition of the cost measure admits a few variants, but all of them quite naturally lower bound the corresponding communication cost. The power of this technique comes from a natural direct sum property of information cost, which allows one to easily lower bound $IC(f)$ for certain well-structured functions f . Specifically, when f is a “combination” of n copies of a simpler function g , one can often scale up a lower bound on $IC(g)$ to obtain $IC(f) \geq \Omega(n IC(g))$. The burden then shifts to lower bounding $IC(g)$, and at this stage the rectangle property is invoked, *but on protocols for g , not f* .

A nice consequence of lower bounding $R(f)$ via a lower bound on $IC(f)$ is that one then obtains a *direct sum theorem* for free: that is, we obtain the bound $R(f^n) \geq \Omega(n IC(f))$ as an almost immediate corollary. We shall be more precise about this in Section 2.

1.2 First Contribution: Rectangular versus Informational Methods

It is natural to ask how, quantitatively, these numerous lower bounding techniques relate to one another. One expects the various “rectangular” techniques to relate to one another, and indeed several such results are known [Kla03, LS09, JK10]. Here, we relate the “informational” technique to one of the most powerful rectangular techniques, with respect to randomized communication complexity. To motivate our first theorem, we begin with a sweeping conjecture.

Conjecture 1.1. *The best information complexity lower bound on $R(f)$ is, asymptotically, at least as good as the smooth corruption (a.k.a., smooth rectangle) bound, and hence, at least as good as the corruption, smooth discrepancy and discrepancy bounds.*

We point out that a very recent manuscript of Kerenidis et al. [KLL⁺12] claims to have settled this conjecture (for a natural setting of parameters). Since this work was done independent of theirs, and due to the short interval between this writing and theirs, we shall continue to label the statement as (our) conjecture.

In conjunction with the results of Jain and Klauck [JK10], the above conjecture states that information complexity subsumes just about every other lower bound technique for $R(f)$. All of these lower bound techniques involve a choice of an input distribution. What we are able to prove is a special case of the conjecture: the case when the input distributions involved are rectangular.³ The statement below is somewhat informal and neither fully detailed nor fully general: a precise version appears as Theorem 3.1.

Theorem 1.2. *Let ρ be a rectangular input distribution for a communication problem $f : \{-1, 1\}^n \times \{-1, 1\}^n \rightarrow \{-1, 1\}$. Then, with respect to ρ , for small enough errors ε , the information complexity bound $IC_\varepsilon^\rho(f)$ is asymptotically as good as the smooth corruption bound $scb_{400\varepsilon, \varepsilon}^\rho(f)$ with error parameter 400ε and perturbation parameter ε . That is, we have $IC_\varepsilon^\rho(f) = \Omega(scb_{400\varepsilon, \varepsilon}^\rho(f))$.*

²Jain and Klauck [JK10] used the term “smooth rectangle bound”, but we shall prefer the more descriptive term “corruption” to “rectangle” throughout this article.

³Some authors use the term “product distribution” for what we call rectangular distributions.

Precise definitions of the terms in the above theorem are given in Section 2. We note that a recent manuscript [BW11] lower bounds information complexity by discrepancy, a result that is similar in spirit to ours. This result is incomparable with ours, because on the one hand discrepancy is a weaker technique than corruption, but on the other hand there is no restriction on the input distribution.

We remark that our proof of Theorem 1.2 uses only elementary combinatorial and information theoretic arguments, and proceeds along intuitive lines. Accordingly, we believe that it remains of independent interest, despite the very recent claim to a stronger result by Kerenidis et al. [KLL⁺12].

1.3 Second Contribution: Information Complexity of Orthogonality and Gap-Hamming

The APPROXIMATE-ORTHOGONALITY problem is a communication problem defined on inputs in $\{-1, 1\}^n \times \{-1, 1\}^n$ by the Boolean function

$$\text{ORT}_{b,n}(x, y) = \begin{cases} 1, & \text{if } |\langle x, y \rangle| \leq b\sqrt{n}, \\ -1, & \text{otherwise.} \end{cases}$$

Here, b is to be thought of as a constant parameter. This problem arose naturally in Sherstov's work on the Gap-Hamming Distance problem [She11a]. This latter problem is defined as follows:

$$\text{GHD}_n(x, y) = \begin{cases} -1, & \text{if } \langle x, y \rangle \leq -\sqrt{n}, \\ 1, & \text{if } \langle x, y \rangle \geq \sqrt{n}. \end{cases}$$

The Gap-Hamming problem has attracted plenty of attention over the last decade, starting from its formal introduction in Indyk and Woodruff [IW03] in the context of data stream lower bounds, leading up to a recent flurry of activity that has produced three different proofs [CR11, Vid11, She11a] of an optimal lower bound $R(\text{GHD}_n) = \Omega(n)$. In some recent work, Woodruff and Zhang [WZ11] identify a need for strong lower bounds on $\text{IC}(\text{GHD})$, to be used in direct sum results. We now attempt to address such a lower bound.

At first sight, these problems appear to be ideally suited for a lower bound via information complexity: they are quite naturally combinations of n independent communication problems, each of which gives Alice and Bob a single input bit each. One feels that the uniform input distribution ought to be hard for them for the intuitive reason that a successful protocol cannot afford to ignore $\omega(\sqrt{n})$ of the coordinates of x and y , and must therefore convey $\Omega(1)$ information per coordinate for at least $\Omega(n)$ coordinates. However, turning this intuition into a formal proof is anything but simple.

Here, we prove an optimal $\Omega(n)$ lower bound on $\text{IC}(\text{ORT})$ under the uniform input distribution. This is a consequence of Theorem 1.2 above, but there turns out to be a surprising amount of work in lower bounding $\text{scb}(\text{ORT})$ under the uniform distribution. Our theorem involves the tail of the standard normal distribution, which we denote by “tail”:

$$\text{tail}(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-x^2/2} dx.$$

We also reserve μ for the uniform distribution on $\{-1, 1\}^n \times \{-1, 1\}^n$.

Theorem 1.3. *Let b be a sufficiently large constant. Then, the corruption bound $\text{cb}_\theta^{1,\mu}(\text{ORT}_{b,n}) = \Omega(n)$, for $\theta = \text{tail}(2.01b)$. Hence, by Theorem 1.2, we have $\text{IC}_{\theta/400}^\mu(\text{ORT}_{b,n}) = \Omega(n)$.*

Again, precise definitions of the terms in the above theorem are given in Section 2 and the proof of the theorem appears in Section 4. As it turns out, a slight strengthening of the parameter θ in the above theorem would give us the result $\text{IC}_{\theta'}^\mu(\text{GHD}_n) = \Omega(n)$. This is because the following result—stated somewhat imprecisely for now—connects the two problems.

Theorem 1.4. *Let b be a sufficiently large constant and let $\theta = \text{tail}(1.99b)$. Then, we have $\text{scb}_{400\theta, \theta}^\mu(\text{GHD}_n) = \Omega(\text{cb}_{400\theta}^{1, \mu}(\text{ORT}_{b, n})) - O(\sqrt{n})$. By Theorem 1.2, we then have $\text{IC}_\theta^\mu(\text{GHD}_n) = \Omega(\text{cb}_{400\theta}^{1, \mu}(\text{ORT}_{b, n})) - O(\sqrt{n})$.*

We note that Chakrabarti and Regev [CR11] state that their lower bound technique for $\text{R}(\text{GHD}_n)$ can be captured within the smooth rectangle bound framework. While this is true in spirit, there is a significant devil in the details, and their technique does not yield a good lower bound on $\text{scb}_{\varepsilon, \delta}^\mu(\text{GHD}_n)$ for the *uniform* distribution μ . We explain more in Section 4.

These theorems suggest a natural follow-up conjecture that we leave open.

Conjecture 1.5. *There exists a constant ε such that $\text{IC}_\varepsilon^\mu(\text{GHD}_n) = \Omega(n)$.*

1.4 Direct Sum

A direct sum theorem states that solving m independent instances of a problem requires about m times the resources that solving a single instance does. It could apply to a number of models of computation, with “resources” interpreted appropriately. For our model of two-party communication, it works as follows. For a function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{-1, 1\}$, let $f^m : \mathcal{X}^m \times \mathcal{Y}^m \rightarrow \{-1, 1\}^m$ denote the function given by

$$f^m(x_1, \dots, x_m, y_1, \dots, y_m) = (f(x_1, y_1), \dots, f(x_m, y_m)).$$

Notice that f^m is not a Boolean function. We will define $\text{R}(f^m)$ to be the randomized communication complexity of the task of outputting a vector (z_1, \dots, z_m) such that for each $i \in [m]$, we have $f(x_i, y_i) = z_i$ with high probability. Then, a direct sum theorem for randomized communication complexity would say that $\text{R}(f^m) = \Omega(m \cdot \text{R}(f))$. Whether or not such a theorem holds for a general f is a major open question in the field.

Information complexity, by its very design, provides a natural approach towards proving a direct sum theorem. Indeed, this was the original motivation of Chakrabarti et al. [CSWY01] in introducing information complexity; they proved a direct sum theorem for randomized *simultaneous-message* and *one-way* complexity, for functions f satisfying a certain “robustness” condition. Still using information complexity, Jain et al. [JRS03] proved a direct sum theorem for bounded-round randomized complexity, when f is hard under a product distribution. Recently, Barak et al. [BBCR10] used information complexity, together with a *protocol compression* approach, to mount the strongest attack yet on the direct sum question for $\text{R}(f)$, for fairly general f : they show that $\text{R}(f^m) \approx \Omega(\sqrt{m} \cdot \text{R}(f))$, where the “ \approx ” ignores logarithmic factors.

One consequence of our work here is a simple proof of a direct sum theorem for randomized communication complexity for functions whose hardness is captured by a smooth corruption bound (which in turn subsumes corruption, discrepancy and smooth discrepancy [JK10]) under a rectangular distribution. This includes the well-studied INNER-PRODUCT function, and thanks to our Theorem 1.3, it also includes ORT. Should Conjecture 1.1 be shown to hold, we could remove the rectangularity constraint altogether and capture additional important functions such as DISJOINTNESS, whose hardness seems to be captured only by considering corruption under a non-rectangular distribution.

We note that the protocol compression approach [BBCR10] gives a strong direct sum result for distributional complexity under rectangular distributions, but still not as strong as ours because their result contains a not-quite-benign polylogarithmic factor. We say more about this in Section 4.

Comparison with Direct Product. Other authors have considered a related, yet different, concept of direct *product* theorems. A strong direct product theorem (henceforth, SDPT) says that computing f^m with a correctness probability as small as $2^{-\Omega(m)}$ —but more than the trivial guessing bound—requires $\Omega(m \text{R}(f))$ communication, where “correctness” means getting *all* m coordinates of the output right. It is known that SDPTs do not hold in all situations [Sha03], but do hold for (generalized) discrepancy [LSv08, She11b], an

especially important technique in lower bounding quantum communication. A recent manuscript offers an SDPT for bounded-round randomized communication [JPY12].

Although strong direct product theorems appear stronger than direct sum theorems,⁴ they are in fact incomparable. A protocol could conceivably achieve low error on each coordinate of $f^m(x_1, \dots, x_m, y_1, \dots, y_m)$ while also having zero probability of getting the entire m -tuple right.

2 Preliminaries

Consider a function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are nonempty finite sets. Although we will develop some initial theory under this general setting, it will be useful to keep in mind the important special case $\mathcal{X} = \mathcal{Y} = \{-1, 1\}^n$ and $\mathcal{Z} = \{-1, 1\}$. We can interpret such a function f as a *communication problem* wherein Alice receives an input $x \in \mathcal{X}$, Bob receives an input $y \in \mathcal{Y}$, and the players must communicate according to a *protocol* P to come up with a value $z \in \mathcal{Z}$ that is hopefully equal to $f(x, y)$. The sequence of messages exchanged by the players when executing P on input (x, y) is called the *transcript* of P on that input, and denoted $P(x, y)$. We require that the transcript be a sequence of bits, and end with (a binary encoding of) the agreed-upon output. We denote the output corresponding to a transcript t by $\text{out}(t)$: thus, the output of P on input (x, y) is $\text{out}(P(x, y))$.

Our protocols will, in general, be randomized protocols with a public coin as well as a private coin for each player. When we disallow the public coin, we will explicitly state that the protocol is private-coin. Notice that $P(x, y)$ is a random string, even for a fixed input (x, y) . For a real quantity $\varepsilon \geq 0$, we say that P computes f with ε error if $\Pr[\text{out}(P(x, y)) \neq f(x, y)] \leq \varepsilon$, the probability being with respect to the randomness used by P and the input distribution. We define the cost of P to be the worst case length of its transcript, $\max |P(x, y)|$, where we maximize over all inputs (x, y) and over all possible outcomes of the coin tosses in P . Finally, the ε -error randomized communication complexity of f is defined by

$$R_\varepsilon(f) = \min\{\text{cost}(P) : P \text{ computes } f \text{ with error } \varepsilon\}.$$

In case $\mathcal{Z} = \{-1, 1\}$, we also put $R(f) = R_{1/3}(f)$.

For random variables A, B, C , we use notations of the form $H(A)$, $H(A | C)$, $H(AB)$, $I(A : B)$, and $I(A : B | C)$ to denote entropy, conditional entropy, joint entropy, mutual information, and conditional mutual information respectively. For discrete probability distributions λ, μ , we use $D_{\text{KL}}(\lambda \parallel \mu)$ to denote the relative entropy (a.k.a., informational divergence or Kullback-Leibler divergence) from λ to μ using logarithms to the base 2. These standard information theoretic concepts are well described in a number of textbooks, e.g., Cover and Thomas [CT06].

Let λ be an input distribution for f , i.e., a probability distribution on $\mathcal{X} \times \mathcal{Y}$. We say that λ is a *rectangular distribution* if we can write it as a tensor product $\lambda = \lambda_1 \otimes \lambda_2$, where λ_1, λ_2 are distributions on \mathcal{X}, \mathcal{Y} respectively. Now consider a general λ and let $(X, Y) \sim \lambda$ be a random input for f drawn from this joint distribution. We define the λ -information-cost of the protocol P to be $\text{icost}^\lambda(P) = I(XY : P(X, Y) | R)$, where R denotes the public randomness used by P . This cost measure gives us a different complexity measure called the ε -error *information complexity* of f , under λ :

$$\text{IC}_\varepsilon^\lambda(f) = \min\{\text{icost}^\lambda(P) : P \text{ computes } f \text{ with error } \varepsilon\}.$$

We note that in the terminology of Barak et al. [BBCR10], the above quantity would be called the *external* information complexity, as opposed to the *internal* one, which is based on the cost function $I(X : P(X, Y), R |$

⁴Some authors interpret “direct sum” as requiring correctness of the entire m -tuple output with high probability. Under this interpretation, direct product theorems indeed subsume direct sum theorems. Our definition of direct sum is arguably more natural, because under our definition, we at least have $R(f^m) = O(m R(f))$ always.

$Y) + I(Y : P(X, Y), R \mid X)$. As noted by them, the two cost measures coincide under a rectangular input distribution. Since our work only concerns rectangular distributions, this internal/external distinction is not important to us.

It is easy to see (and by now well-known) that information complexity under *any* input distribution lower bounds randomized communication complexity.

Fact 2.1. *For every input distribution λ and error ε , we have $R_\varepsilon(f) \geq IC_\varepsilon^\lambda(f)$.*

Proof. Simply observe that $I(XY : P(X, Y) \mid R) \leq H(P(X, Y)) \leq |P(X, Y)|$. \square

2.1 Corruption and Smooth Corruption

We consider a communication problem given by a partial function, $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z} \cup \{*\}$. We say that the function f is undefined on an input $(x, y) \in \mathcal{X} \times \mathcal{Y}$ iff $f(x, y) = *$. For such inputs we say that a protocol P computes f correctly on (x, y) always, irrespective of what P outputs. Therefore, we say that a protocol P computes f with error $\varepsilon \geq 0$ if $\Pr[f(x, y) \neq * \wedge \text{out}(P(x, y)) \neq f(x, y)] \leq \varepsilon$ where, as before, the probability being with respect to the randomness used by P and the input distribution.

Pick a particular $z \in \mathcal{Z}$. A set $S \subseteq \mathcal{X} \times \mathcal{Y}$ is said to be *rectangular* if we have $S = S_1 \times S_2$, where $S_1 \subseteq \mathcal{X}, S_2 \subseteq \mathcal{Y}$. Following Beame et al. [BPSW06], we say that S is ε -error z -monochromatic for f under λ if $\lambda(S \setminus (f^{-1}(z) \cup f^{-1}(*))) \leq \varepsilon \lambda(S)$. We then define

$$\varepsilon\text{-mono}^{z, \lambda}(f) = \max\{\lambda(S) : S \text{ is rectangular and } \varepsilon\text{-error } z\text{-monochromatic}\}, \quad (1)$$

$$\text{cb}_\varepsilon^{z, \lambda}(f) = -\log(\varepsilon\text{-mono}^{z, \lambda}(f)), \quad (2)$$

$$\text{scb}_{\varepsilon, \delta}^{z, \lambda}(f) = \max\{\text{cb}_\varepsilon^{z, \lambda}(g) : g \in (\mathcal{Z} \cup \{*\})^{\mathcal{X} \times \mathcal{Y}}, \Pr_{(X, Y) \sim \lambda}[f(X, Y) \neq g(X, Y)] \leq \delta\}. \quad (3)$$

The quantities $\text{cb}_\varepsilon^{z, \lambda}(f)$ and $\text{scb}_{\varepsilon, \delta}^{z, \lambda}(f)$ are called the corruption bound and the smooth corruption bound respectively, under the indicated choice of parameters. In the latter quantity, we refer to ε as the *error parameter* and δ as the *perturbation parameter*. One can go on to define bounds independent of z and λ by appropriately maximizing over these two parameters, but we shall not do that here.

We note that Jain and Klauck [JK10] use somewhat different notation: what we have called scb above is the logarithm of (a slight variant of) the quantity that they call the “natural definition of the smooth rectangle bound” and denote $\widetilde{\text{rec}}$.

What justifies calling these quantities “bounds” is that they can be shown to lower bound $R_{\varepsilon'}(f)$ for sufficiently small $\delta, \varepsilon, \varepsilon'$, under a mild condition on λ . It is clear that $\text{scb}_{\varepsilon, \delta}^{z, \lambda}(f) \geq \text{cb}_\varepsilon^{z, \lambda}(f)$, so we mention only the stronger result, that involves the smooth corruption bound.

Fact 2.2 (Jain and Klauck [JK10]). *Let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z} \cup \{*\}$, $z \in \mathcal{Z}$ and distribution λ on $\mathcal{X} \times \mathcal{Y}$ be such that $\lambda(f^{-1}(z)) \geq 1/3$. Then there is an absolute constant $c > 0$ such that, for a sufficiently small constant ε , we have $R_\varepsilon(f) \geq c \cdot \text{scb}_{2\varepsilon, \varepsilon/2}^{z, \lambda}(f)$.* \square

The constant $1/3$ above is arbitrary and can be parametrized, but we avoid doing this to keep things simple. The proof of the above fact is along the expected lines: an application of (the easy direction of) Yao’s minimax lemma, followed by a straightforward estimation argument applied to the rectangles of the resulting deterministic protocol. Note that we never have to involve the linear-programming-based smooth rectangle bound as defined by Jain and Klauck.

3 Information Complexity versus Corruption

We are now in a position to tackle our first theorem.

Theorem 3.1 (Precise restatement of Theorem 1.2). *Suppose we have a function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z} \cup \{*\}$, a rectangular distribution ρ on $\mathcal{X} \times \mathcal{Y}$, and $z \in \mathcal{Z}$ satisfying $\rho(f^{-1}(z)) \geq 3/20$. Let $\varepsilon, \varepsilon'$ be reals with $0 \leq 384\varepsilon \leq \varepsilon' < 1/4$. Then*

$$\text{IC}_\varepsilon^\rho(f) \geq \frac{1}{400} \text{scb}_{\varepsilon', \varepsilon}^{z, \rho}(f) - \frac{1}{50} = \Omega(\text{scb}_{\varepsilon', \varepsilon}^{z, \rho}(f)) - O(1).$$

To prove this, we first consider a notion that we call the *distortion* of a transcript of a communication protocol. Let ρ be an input distribution for a communication problem, let P be a protocol for the problem, and let t be a transcript of P . We define $\sigma_t = \sigma_t(\rho)$ to be the distribution $(\rho \mid P(X, Y) = t)$. We think of the relative entropy $D_{\text{KL}}(\sigma_t \parallel \rho)$ as a distortion measure for t : intuitively, if t conveys little information about the inputs, then this distortion should be low. The following lemma makes this intuition precise. Notice that it does not assume that ρ is rectangular.

For the remainder of this section, to keep the notation simple while handling partial functions, we write $g(x, y) \neq z$ to actually denote the event $g(x, y) \neq z \wedge g(x, y) \neq *$ for $z \in \mathcal{Z}$, unless specified otherwise.

Lemma 3.2. *Let P be a private-coin protocol that computes $g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z} \cup \{*\}$ with error $\varepsilon < 1/500$. Let $z \in \mathcal{Z}$ and let ρ be an arbitrary distribution on $\mathcal{X} \times \mathcal{Y}$ with $\rho(g^{-1}(z)) \geq 3/20 - 1/500$. Then, there exists a (“low-distortion”) transcript t of P such that*

$$\text{out}(t) = z, \tag{4}$$

$$D_{\text{KL}}(\sigma_t \parallel \rho) \leq 50 \text{icost}^\rho(P), \text{ and} \tag{5}$$

$$\Pr[g(X, Y) \neq z \mid T = t] \leq 8\varepsilon, \tag{6}$$

where $(X, Y) \sim \rho$ and $T = P(X, Y)$.

Proof. Let τ denote the distribution on transcripts given by $P(X, Y)$. By basic results in information theory [CT06], we have

$$\text{icost}^\rho(P) = I(XY : T) = \mathbb{E}_{T \sim \tau} [D_{\text{KL}}(\sigma_T \parallel \rho)].$$

Consider a random choice of t according to τ . By Markov’s inequality, conditions (5) and (6) fail with probability at most $1/50$ and $1/8$ respectively. By the lower bound on $\rho(g^{-1}(z))$, condition (4) fails with probability at most $17/20 + 1/500 + \varepsilon$. Since $\varepsilon \leq 1/500$, and $1/8 + 1/50 + 17/20 + 1/500 + 1/500 < 1$, it follows that there exists a choice of t satisfying all three conditions. \square

Property 6 in the above lemma should be interpreted as a low-error guarantee for the transcript t . We now argue that the existence of such a transcript implies the existence of a “large” low-corruption rectangle, provided the input distribution ρ is rectangular: this is the only point in the proof that uses rectangularity. One has to be careful with the interpretation of “large” here: it means large under σ_t , and not ρ . However, later on we will add in the low-distortion guarantee of Lemma 3.2 to conclude largeness under ρ as well.

Lemma 3.3. *Let t be a transcript of a private-coin protocol P for $g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z} \cup \{*\}$. Let ρ be a rectangular distribution on $\mathcal{X} \times \mathcal{Y}$, $z \in \mathcal{Z}$, $(X, Y) \sim \rho$, $T = P(X, Y)$, and $\varepsilon \geq 0$. Suppose*

$$\Pr[g(X, Y) \neq z \mid T = t] \leq \varepsilon, \tag{7}$$

then there exists a rectangle $L \subseteq \mathcal{X} \times \mathcal{Y}$ such that

$$\sigma_t(L) \geq 9/16, \text{ and} \tag{8}$$

$$\Pr[g(X, Y) \neq z \mid (X, Y) \in L] \leq 16\varepsilon. \tag{9}$$

Proof. By the rectangle property for private-coin protocols [BJKS04, Lemma 6.7], there exist mappings $q_1 : \mathcal{X} \rightarrow [0, 1], q_2 : \mathcal{Y} \rightarrow [0, 1]$ such that $\Pr[T = t \mid X = x, Y = y] = q_1(x)q_2(y)$.

Let τ denote the distribution of T . We can rewrite the condition (7) as

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y} : g(x, y) \neq z} q_1(x)q_2(y)\rho(x, y) \leq \varepsilon \tau(t). \quad (10)$$

Consider the set \mathcal{A} of rows whose contribution to the left hand side of (10) is “low,” i.e.,

$$\mathcal{A} = \left\{ x \in \mathcal{X} : \sum_{y : g(x, y) \neq z} q_2(y)\rho(x, y) \leq 4\varepsilon \sum_y q_2(y)\rho(x, y) \right\}.$$

Then, by a Markov-inequality-style argument, we have $\Pr[X \in \mathcal{A} \mid T = t] \geq \frac{3}{4}$.

Similarly, consider the following set \mathcal{B} of columns (notice that we sum over only $x \in \mathcal{A}$):

$$\mathcal{B} = \left\{ y \in \mathcal{Y} : \sum_{x \in \mathcal{A} : g(x, y) \neq z} \rho(x, y) \leq 16\varepsilon \sum_{x \in \mathcal{A}} \rho(x, y) \right\}.$$

We now claim that the rectangle $\mathcal{A} \times \mathcal{B}$ has the desired properties.

From the definition of \mathcal{B} , it follows that for all $y \in \mathcal{B}$, $\Pr[g(X, y) \neq z \mid X \in \mathcal{A}] \leq 16\varepsilon$. Therefore, we have $\Pr[g(X, Y) \neq z \mid (X, Y) \in \mathcal{A} \times \mathcal{B}] \leq 16\varepsilon$ and hence, the rectangle $\mathcal{A} \times \mathcal{B}$ satisfies condition (9).

Since we know that $\Pr[X \in \mathcal{A} \mid T = t] \geq 3/4$, to prove that $\Pr[(X, Y) \in \mathcal{A} \times \mathcal{B} \mid T = t] \geq 9/16$ we will first show that the columns in \mathcal{B} have significant “mass” in \mathcal{A} using averaging arguments.

Claim 3.4. We have $\sum_{x \in \mathcal{A}} \sum_{y \in \mathcal{B}} q_2(y)\rho(x, y) \geq \frac{3}{4} \sum_{x \in \mathcal{A}} \sum_{y \in \mathcal{Y}} q_2(y)\rho(x, y)$.

Proof. Assume not. Then $\sum_{x \in \mathcal{A}} \sum_{y \in \mathcal{Y} \setminus \mathcal{B}} q_2(y)\rho(x, y) \geq \frac{1}{4} \sum_{x \in \mathcal{A}} \sum_{y \in \mathcal{Y}} q_2(y)\rho(x, y)$. Therefore,

$$\begin{aligned} \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{A} : g(x, y) \neq z} q_2(y)\rho(x, y) &\geq \sum_{y \in \mathcal{Y} \setminus \mathcal{B}} q_2(y) \sum_{x \in \mathcal{A} : g(x, y) \neq z} \rho(x, y) \\ &> 16\varepsilon \sum_{y \in \mathcal{Y} \setminus \mathcal{B}} q_2(y) \sum_{x \in \mathcal{A}} \rho(x, y) \quad (\text{by def of } \mathcal{B}) \\ &\geq 4\varepsilon \sum_{y \in \mathcal{Y}} q_2(y) \sum_{x \in \mathcal{A}} \rho(x, y), \end{aligned}$$

which contradicts the definition of \mathcal{A} . \square

Recall that ρ is a rectangular distribution. Suppose η_1 and η_2 are its marginals, i.e., $\rho(x, y) = \eta_1(x)\eta_2(y)$. We now observe that the fraction $\sum_{y \in \mathcal{B}} q_2(y)\rho(x, y) / \sum_{y \in \mathcal{Y}} q_2(y)\rho(x, y)$ is the same for all $x \in \mathcal{X}$. We have

$$\frac{\sum_{y \in \mathcal{B}} q_2(y)\rho(x, y)}{\sum_{y \in \mathcal{Y}} q_2(y)\rho(x, y)} = \frac{\sum_{y \in \mathcal{B}} q_2(y)\eta_1(x)\eta_2(y)}{\sum_{y \in \mathcal{Y}} q_2(y)\eta_1(x)\eta_2(y)} = \frac{\eta_1(x) \sum_{y \in \mathcal{B}} q_2(y)\eta_2(y)}{\eta_1(x) \sum_{y \in \mathcal{Y}} q_2(y)\eta_2(y)} = \frac{\sum_{y \in \mathcal{B}} q_2(y)\eta_2(y)}{\sum_{y \in \mathcal{Y}} q_2(y)\eta_2(y)},$$

which is indeed independent of x . Denote this fraction by κ . With the above observation and claim 3.4, we

can conclude that $\kappa \geq 3/4$. We can now prove that the rectangle $\mathcal{A} \times \mathcal{B}$ satisfies condition (8) as follows:

$$\begin{aligned}
\sigma_t(\mathcal{A} \times \mathcal{B}) &= \sum_{x \in \mathcal{A}} \sum_{y \in \mathcal{B}} \rho(x, y) q_1(x) q_2(y) / \tau(t) \\
&= \sum_{x \in \mathcal{A}} \eta_1(x) q_1(x) \sum_{y \in \mathcal{B}} q_2(y) \eta_2(y) / \tau(t) \\
&= \sum_{x \in \mathcal{A}} \eta_1(x) q_1(x) \kappa \sum_{y \in \mathcal{Y}} q_2(y) \eta_2(y) / \tau(t) \\
&= \kappa \sum_{x \in \mathcal{A}, y \in \mathcal{Y}} \eta_1(x) q_1(x) q_2(y) \eta_2(y) / \tau(t) \\
&= \kappa \sum_{x \in \mathcal{A}, y \in \mathcal{Y}} \rho(x, y) q_1(x) q_2(y) / \tau(t) \\
&= \kappa \Pr[X \in \mathcal{A} \mid T = t] \geq \frac{3\kappa}{4} \geq \frac{9}{16}. \quad \square
\end{aligned}$$

The proof of our next lemma uses the (classical) Substate Theorem due to Jain, Radhakrishnan and Sen [JRS09]. We state this below in a form that is especially useful for us: it says roughly that if the relative entropy $D_{\text{KL}}(\lambda_1 \parallel \lambda_2)$ is upper bounded, then the events that have significant probability under λ_1 continue to have significant probability under λ_2 .

Fact 3.5 (Substate Theorem [JRS09]). *Let λ_1 and λ_2 be distributions on a set \mathcal{X} with $D_{\text{KL}}(\lambda_1 \parallel \lambda_2) \leq d$, for some positive d . Then, for all $S \subseteq \mathcal{X}$, we have $\lambda_2(S) \geq \lambda_1(S) / 2^{2+2/\lambda_1(S)+2d/\lambda_1(S)}$.* \square

Lemma 3.6. *Let t be a transcript of a private-coin protocol P for $g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z} \cup \{*\}$, and suppose $\text{out}(t) = z \in \mathcal{Z}$. Let ρ be a rectangular distribution on $\mathcal{X} \times \mathcal{Y}$, and $\varepsilon \leq 1$. Then at most one of the following conditions can hold:*

$$D_{\text{KL}}(\sigma_t \parallel \rho) < (\text{cb}_{\varepsilon}^{z, \rho}(g) - 7)/4, \quad (11)$$

$$\Pr[g(X, Y) \neq z \mid T = t] \leq \varepsilon/16, \quad (12)$$

where $(X, Y) \sim \rho$, $T = P(X, Y)$, and $\sigma_t = (\rho \mid T = t)$.

Proof. Suppose condition (12) holds. Then Lemma 3.3 implies that there exists a rectangle L such that $\sigma_t(L) \geq 9/16$ and $\Pr[g(X, Y) \neq z \mid (X, Y) \in L] \leq \varepsilon$. The latter condition may be rewritten as $\rho(L \setminus (g^{-1}(z) \cup g^{-1}(**))) \leq \varepsilon \rho(L)$, i.e., L is ε -error z -monochromatic for g under ρ .

Suppose (11) also holds. Then, by the Substate Theorem, for every subset $S \subseteq \mathcal{X} \times \mathcal{Y}$, we have

$$\rho(S) \geq \frac{\sigma_t(S)}{2^{2+2/\sigma_t(S)+2d/\sigma_t(S)}},$$

where $d = D_{\text{KL}}(\sigma_t \parallel \rho)$. Taking S to be the above rectangle L , and noting that $\sigma_t(L) \geq 1/2$, we have

$$\rho(L) \geq \frac{1}{2^{7+4d}} > \frac{1}{2^{\text{cb}_{\varepsilon}^{z, \rho}(g)}}.$$

Since L is ε -error z -monochromatic, the definition of the corruption bound tells us that $\text{cb}_{\varepsilon}^{z, \rho}(g) \leq -\log \rho(L)$, which contradicts the above inequality. \square

Proof of Theorem 3.1. Suppose, to the contrary, that $\text{IC}_{\varepsilon}^{\rho}(f) \leq \text{sbc}_{\varepsilon', \varepsilon}^{z, \rho}(f)/400 - 1/50$. Let P^* be a protocol for f achieving the ε -error information cost under ρ . By a standard averaging argument, we may fix the public randomness of P^* to obtain a private-coin protocol P that computes f with error 2ε , and has $\text{icost}^{\rho}(P) \leq 2\text{icost}^{\rho}(P^*)$. Let g be the function achieving the maximum in Eq. (3), the definition of the

smooth corruption bound, with error parameter ε' and perturbation parameter ε . Then $\text{scb}_{\varepsilon',\varepsilon}^{z,P}(f) = \text{cb}_{\varepsilon'}^{z,P}(g)$ and P computes g with error $3\varepsilon \leq 1/500$. Furthermore,

$$\rho(g^{-1}(z)) \geq \rho(f^{-1}(z)) - \Pr_{(X,Y) \sim \rho} [f(X,Y) \neq g(X,Y)] \geq 3/20 - \varepsilon > 3/20 - 1/500.$$

By Lemma 3.2, there exists a transcript t of P satisfying conditions (4), (5), and (6). The right hand side of (5) is at most $100 \text{icost}^P(P^*) < (\text{scb}_{\varepsilon',\varepsilon}^{z,P}(f) - 7)/4 = (\text{cb}_{\varepsilon'}^{z,P}(g) - 7)/4$ and the right hand side of (6) is at most $24\varepsilon \leq \varepsilon'/16$.

Therefore, conditions (11) and (12) in Lemma 3.6 are *both* satisfied, while $\text{out}(t) = z$ and ρ is rectangular, which contradicts that lemma. \square

4 The Information Complexity of Orthogonality and Gap-Hamming

We now tackle Theorems 1.3 and 1.4. Since these results are closely connected with a few recent works, and are both conceptually and technically interesting in their own right, we begin by discussing why they take so much additional work.

For the remainder of this paper, μ_n will denote the uniform distribution on $\{-1, 1\}^n \times \{-1, 1\}^n$. We will almost always drop the subscript n and simply use μ .

4.1 The Orthogonality Problem

The first thing to address is why the information complexity of these problems is not already lower bounded by an existing general result of Barak et al. [BBCR10].

The Barak-Braverman-Chen-Rao Approach. The protocol compression technique given by Barak et al. for *rectangular* distributions relates information complexity under such distributions to communication complexity in what seems like a near-optimal way. Why then are we not happy with their result? To understand this, consider a protocol P for $\text{ORT}_{1/4,n}$ with communication cost c , error ε (for some sufficiently small constant ε) and information cost d , under the uniform distribution μ . Their compression result would compress P to a 2ε -error ORT protocol P^* with

$$\text{cost}(P^*) = O\left(\frac{d \log(c/\varepsilon)}{\varepsilon^2}\right).$$

By the distributional complexity lower bound for $\text{ORT}_{1/4,n}$ [She11a], we have $\text{cost}(P^*) = \Omega(n)$. *However, this does not imply $d = \Omega(n)$ or even $d = \Omega(n/\text{polylog}(n))$!* In particular, we may have the weird situation that $d = O(1)$ and $c = 2^{\Omega(n)}$. Thus, our lower bound for $\text{IC}(\text{ORT}_{b,n})$ is in fact a strong result, far from what follows from prior work.

A Word About Our Approach. Turning to *our* proof for a moment, we now see that we need to lower bound $\text{cb}^\lambda(\text{ORT}_{b,n})$ for a *rectangular* λ . We make the most natural choice, picking $\lambda = \mu$, the uniform input distribution. Our proof is then heavily inspired by two recent proofs of an optimal $\Omega(n)$ lower bound on $\text{R}(\text{GHD}_n)$, namely those of Chakrabarti and Regev [CR11], and Sherstov [She11a]. At the heart of our proof is the following anti-concentration lemma, which says that when pairs (x,y) are randomly drawn from a large rectangle in $\{-1, 1\}^n \times \{-1, 1\}^n$, the inner product $\langle x,y \rangle$ cannot be too sharply concentrated around zero.

Lemma 4.1 (Anti-concentration). *Let n be sufficiently large, let $b \geq 66$ be a constant, and let $\varepsilon = \text{tail}(2.01b)$. Then there exists $\delta > 0$ such that for all $A, B \subseteq \{-1, 1\}^n$ with $\min\{|A|, |B|\} \geq 2^{n-\delta n}$, we have*

$$\Pr_{(X,Y) \in_R A \times B} [\langle X, Y \rangle \notin [-b\sqrt{n}, b\sqrt{n}]] \geq \varepsilon, \quad (13)$$

where “ \in_R ” denotes “is chosen uniformly at random from”.

The proof of this anti-concentration lemma has several technical steps, and we give this proof in Section 5. Below, we prove Theorem 1.3 using this lemma, and then discuss what is new about this lemma.

Theorem 4.2 (Precise restatement of Theorem 1.3). *Let $b \geq 1/5$ be a constant. Then $\text{cb}_\theta^{1,\mu}(\text{ORT}_{b,n}) = \Omega(n)$, for $\theta = \text{tail}(2.01 \max\{66, b\})$. Hence, we have $\text{IC}_{\theta/400}^\mu(\text{ORT}_{b,n}) = \Omega(n)$.*

Proof. We first estimate the corruption bound. Let δ be the constant whose existence is guaranteed by Lemma 4.1. For $b \geq 66$, Eq. (13) states precisely that θ -mono $^{1,\mu}(\text{ORT}_{b,n}) \leq 2^{-\delta n}$. Thus, it follows that $\text{cb}_\theta^{1,\mu}(\text{ORT}_{b,n}) \geq \delta n = \Omega(n)$. For $b < 66$, we note that

$$\Pr_{(X,Y) \in_R A \times B} [\langle X, Y \rangle \notin [-b\sqrt{n}, b\sqrt{n}]] \geq \Pr_{(X,Y) \in_R A \times B} [\langle X, Y \rangle \notin [-66\sqrt{n}, 66\sqrt{n}]],$$

for any $A, B \subseteq \{-1, 1\}^n$. Therefore, using Lemma 4.1 as before, we can conclude that $\text{cb}_\theta^{1,\mu}(\text{ORT}_{b,n}) = \Omega(n)$ for $\theta = \text{tail}(2.01 \times 66)$.

To lower bound the information complexity, we first note that

$$\text{scb}_{\theta, \theta/400}^{1,\mu}(\text{ORT}_{b,n}) \geq \text{scb}_{\theta, 0}^{1,\mu}(\text{ORT}_{b,n}) = \text{cb}_\theta^{1,\mu}(\text{ORT}_{b,n}) = \Omega(n).$$

Since $b \geq 1/5$, standard estimates of the tail of a binomial distribution give us that $\mu(\text{ORT}_{b,n}^{-1}(1)) > 3/20$ for large enough n . Further, we have $\theta = \text{tail}(2.01 \max\{66, b\}) < 1/4$. Applying Theorem 3.1, we conclude that $\text{IC}_{\theta/400}^\mu(\text{ORT}_{b,n}) = \Omega(n)$. \square

We now address why the approaches in two recent works do not suffice to prove Lemma 4.1.

The Sherstov Approach. At first glance, Lemma 4.1 may appear to be essentially Sherstov’s Theorem 3.3, but it is not! Sherstov’s theorem is a special case of ours that fixes $b = 1/4$, and the smallness of that choice is crucial to Sherstov’s proof. In particular, his proof does not work once $b > 1$. In order to connect ORT to GHD, however, we need this anti-concentration with b being a *large* constant. Looking ahead a bit, this is because we need the upper bound in Eq. (16) to be tight enough.

The reason that Sherstov’s approach requires b to be small is technical, but here is a high-level overview. He relies on an inequality of Talagrand (which appears as [She11a, Fact 2.2]) which states that the projection of a random vector from $\{-1, 1\}^n$ onto a linear subspace $V \subseteq \mathbb{R}^n$ is sharply concentrated around $\sqrt{\dim V}$, which is at most \sqrt{n} . Once $b > 1$, this sharp concentration works *against* his approach and, in particular, fails to imply anti-concentration of $\langle X, Y \rangle$ in $[-b\sqrt{n}, b\sqrt{n}]$, which is now too large an interval.

The Chakrabarti-Regev Approach. At second glance, Lemma 4.1 may appear to be a variant of the “correlation inequality” (Theorem 3.5 and Corollary 3.8) of Chakrabarti and Regev. This is true to an extent, but crucially our lemma is not a corollary of that correlation inequality, which we state below.

Fact 4.3 (Equivalent to Corollary 3.8 of [CR11]). *Let n be sufficiently large, and let $b > 0$ and $\varepsilon > 0$ be constants. Then there exists $\delta > 0$ such that for all $A, B \subseteq \{-1, 1\}^n$ with $\min\{|A|, |B|\} \geq 2^{n-\delta n}$, we have*

$$v_b(A \times B) \geq (1 - \varepsilon)\mu(A \times B), \quad (14)$$

where $v_b = \frac{1}{2}(\xi_{-2b/\sqrt{n}} + \xi_{2b/\sqrt{n}})$ and ξ_p is the distribution of $(x, y) \in \{-1, 1\}^n \times \{-1, 1\}^n$ where we pick $x \in_R \{-1, 1\}^n$ and choose y by flipping each coordinate of x independently with probability $(1 - p)/2$.

The above is also an anti-concentration statement about inner products in a large rectangle. One might therefore hope to use it to prove Lemma 4.1 by showing that one kind of anti-concentration implies the other for “counting” reasons. That is, one might hope that every large set $S \subseteq \{-1, 1\}^n \times \{-1, 1\}^n$ that satisfies an inequality like (14) also satisfies one like (13).

But this is not the case. Consider the set $S = S_0 \cup S_{2b}$ where S_0 is any subset of $2^{2n-\delta n}$ inputs such that for all $(x, y) \in S_0$ we have $\langle x, y \rangle = 0$, and S_{4b} is any subset of $(\varepsilon/2)|S_0|$ inputs such that for all $(x, y) \in S_{4b}$ we have $\langle x, y \rangle = 4b\sqrt{n}$. Then, by construction, we have $\Pr_{(x,y) \in RS} [\langle x, y \rangle \notin [-b\sqrt{n}, b\sqrt{n}]] \leq \varepsilon/2 < \varepsilon$, so S does not satisfy an inequality like (13). However, for several choices of ε and b , it does satisfy the analogue of inequality (14): a short calculation shows that $v_b(S) \geq \frac{1}{2}\xi_{2b/\sqrt{n}}(S_{4b}) \geq \frac{1}{2}\varepsilon e^{5b^2} \mu(S) \geq \mu(S)$.

Thus, even given Fact 4.3, we still need to use the rectangularity of S to prove Lemma 4.1. It is this need to use rectangularity carefully that leads to the longish technical proof to follow, in Section 5.

4.2 The Gap-Hamming Problem

We now address the issue of proving a strong lower bound on $\text{IC}^\mu(\text{GHD})$. As before, we first note why existing methods do not imply an $\Omega(n)$ lower bound, and then give our approach. We stress that our approach is, at this point, a *program only* and stops short of settling Conjecture 1.5, i.e., proving that $\text{IC}^\mu(\text{GHD}) = \Omega(n)$.

Previous Approaches. The orthogonality problem ORT is intimately related to the Gap-Hamming Distance problem GHD . This was first noted by Sherstov, who used an ingenious technique to prove that $R(\text{GHD}_n) = \Omega(n)$ based on his lower bound $R(\text{ORT}_{1/4,n}) = \Omega(n)$. He gave a reduction from ORT to GHD wherein a protocol for GHD was called *twice* to obtain a protocol for ORT . But this style of reduction does not yield a relation between *information* complexities, and so the lower bound on $\text{IC}^\mu(\text{ORT})$ in Theorem 4.2 does not translate into a lower bound on $\text{IC}^\mu(\text{GHD})$.

The Chakrabarti-Regev proof [CR11] of the same bound $R(\text{GHD}_n) = \Omega(n)$ introduces a technique that they call corruption-with-jokers which in turn is subsumed by what Jain and Klauck [JK10] have called the “smooth rectangle bound.” In fact, Jain and Klauck define two variants of the smooth rectangle bound: a linear-programming-based variant that they denote *srec*, and a “natural” variant that they denote $\widetilde{\text{srec}}$. It is the former variant that subsumes the Chakrabarti-Regev technique, whereas our work here corresponds to the latter variant.

Jain and Klauck do give a pair of translation lemmas, showing that the two variants are asymptotically equivalent up to some changes in parameters. Therefore, the Chakrabarti-Regev approach does yield a lower bound on $\text{scb}^\lambda(\text{GHD}_n)$, but the distribution λ that comes out of applying the appropriate translation lemma is non-rectangular. Therefore, we cannot apply Theorem 3.1.

Furthermore, even granting Conjecture 1.1 (as claimed by Kerenidis et al. [KLL⁺12]), this line of reasoning will only lower bound $\text{IC}^\lambda(\text{GHD})$ for an artificial distribution λ , and will not lower bound $\text{IC}^\mu(\text{GHD})$.

Our Approach. Our idea is that, for large b , the function GHD_n is at least as “hard” as a function that is “close” to $\text{ORT}_{b,n}$, under a uniform input distribution. To be precise, we have the following connection between GHD and ORT . Recall that μ_n is the uniform distribution on $\{-1, 1\}^n \times \{-1, 1\}^n$.

Theorem 4.4 (Precise restatement of Theorem 1.4). *Let n be sufficiently large, let $b \geq 100$ be a constant, and let $\text{tail}(1.99b) \leq \theta \leq 1/1600$. Let $n' = n + \frac{1}{2}(1.99b - 1)\sqrt{n}$. Then, we have*

$$\text{scb}_{400\theta, \theta}^{1, \mu_n}(\text{GHD}_n) = \Omega(\text{cb}_{400\theta}^{1, \mu_{n'}}(\text{ORT}_{b, n'})) - O(\sqrt{n}).$$

Combining this with Theorem 3.1, we then have $\text{IC}_\theta^{\mu_n}(\text{GHD}_n) = \Omega(\text{cb}_{400\theta}^{1, \mu_{n'}}(\text{ORT}_{b, n'})) - O(\sqrt{n})$.

Remark. Suppose we could strengthen Theorem 4.2 by changing the constant 2.01 in that theorem to 1.98, i.e., suppose we had $\text{cb}_\varepsilon^{1,\mu}(\text{ORT}_{b,n}) = \Omega(n)$ with $\varepsilon = \text{tail}(1.98b)$. Then the present theorem would give us $\text{IC}_{\varepsilon/400}^\mu(\text{GHD}_n) = \Omega(n)$, since $\varepsilon/400 > \text{tail}(1.99b)$ for large enough b .

Proof. Put $t = n' - n = \frac{1}{2}(1.99b - 1)\sqrt{n}$. Consider the padding $(x, y) \in \{-1, 1\}^n \mapsto (x', y') \in \{-1, 1\}^{n'}$ defined by $x' = (1, 1, \dots, 1, x)$ and $y' = (-1, -1, \dots, -1, y)$. Then we have $\langle x', y' \rangle = \langle x, y \rangle - t$. Since $b \geq 100$, for $b' := 1.99b$, we have

$$\langle x, y \rangle \in [-\sqrt{n}, b'\sqrt{n}] \implies \langle x', y' \rangle \in [-b'\sqrt{n'}, b'\sqrt{n'}]. \quad (15)$$

Let $h : \{-1, 1\}^n \times \{-1, 1\}^n \rightarrow \{-1, 1\}$ be the partial function defined as follows:

$$h(x, y) = \begin{cases} \text{GHD}_n(x, y), & \text{if } \langle x, y \rangle \leq b'\sqrt{n}, \\ -\text{GHD}_n(x, y), & \text{if } \langle x, y \rangle > b'\sqrt{n}. \end{cases}$$

From (15) and the definition of ORT we can conclude that $\text{ORT}_{b,n'}(x', y') \neq 1 \implies h(x, y) \notin \{1, *\}$ for all $x, y \in \{-1, 1\}^n$. Thus, for any rectangle $R \subseteq \{-1, 1\}^n \times \{-1, 1\}^n$, we have

$$\frac{|\{(x, y) \in R : h(x, y) \notin \{1, *\}\}|}{|R|} \geq \frac{|\{(x', y') \in R' : \text{ORT}_{b,n'}(x', y') \neq 1\}|}{|R'|},$$

where $R' \subseteq \{-1, 1\}^{n'} \times \{-1, 1\}^{n'}$ is the rectangle obtained by padding each $(x, y) \in R$ as above. Therefore, if R is ε -error 1-monochromatic for h under μ_n , then R' is ε -error 1-monochromatic for $\text{ORT}_{b,n'}$ under $\mu_{n'}$. Hence, $\varepsilon\text{-mono}^{1,\mu_n}(h) \leq 2^{2t} \varepsilon\text{-mono}^{1,\mu_{n'}}(\text{ORT}_{b,n'})$ and thus, $\text{cb}_\varepsilon^{1,\mu_n}(h) \geq \text{cb}_\varepsilon^{1,\mu_{n'}}(\text{ORT}_{b,n'}) - 2t$.

By standard estimates of the tail of a binomial distribution [Fel68], we have

$$\Pr_{(X,Y) \sim \mu_n} [h(X, Y) \neq \text{GHD}_n(X, Y)] = \Pr_{(X,Y) \sim \mu_n} [\langle X, Y \rangle > b'\sqrt{n}] \leq \text{tail}(b') = \text{tail}(1.99b). \quad (16)$$

Therefore, $\text{scb}_{\varepsilon,\theta}^{1,\mu_n}(\text{GHD}_n) \geq \text{cb}_\varepsilon^{1,\mu_n}(h) \geq \text{cb}_\varepsilon^{1,\mu_{n'}}(\text{ORT}_{b,n'}) - 2t$ with $\theta \geq \text{tail}(1.99b)$. The proof is now completed by applying Theorem 3.1: for the setting $\varepsilon = 400\theta$, we have $0 \leq 384\theta \leq \varepsilon < 1/4$ and $\mu_n(\text{GHD}_n^{-1}(1)) \geq 3/20$. Therefore, we can conclude

$$\text{IC}_\theta^\mu(\text{GHD}_n) = \Omega(\text{scb}_{\varepsilon,\theta}^{1,\mu_n}(\text{GHD}_n)) - O(1) = \Omega(\text{cb}_\varepsilon^{1,\mu_{n'}}(\text{ORT}_{b,n'})) - O(\sqrt{n}). \quad \square$$

5 Proof of the Anti-Concentration Lemma

Finally, we turn to the most technical part of this work: a proof of our new anti-concentration lemma, stated as Lemma 4.1 earlier.

5.1 Preparatory Work and Proof Overview

Let us begin with some convenient notation. We denote the (density function of the) standard normal distribution on the real line \mathbb{R} by γ . We also denote the standard n -dimensional Gaussian distribution by γ^n . For a set $A \subseteq \mathbb{R}^n$, we denote by $\gamma^n|_A$ the distribution γ^n conditioned on belonging to A . For a distribution P on \mathbb{R}^n , we define its “distance to Gaussianity”, denoted $D_\gamma(P)$ as follows.

$$D_\gamma(P) = D(P \parallel \gamma^n) := \int P(x) \ln \frac{P(x)}{\gamma^n(x)} dx.$$

The latter quantity is the well-known relative entropy for continuous probability distributions, and is the analogue of D_{KL} , which we have used earlier. Note that the logarithm here is to the base e , and not 2 as it was earlier.

Let X, Y be possibly correlated random variables, with density functions P_X and P_Y respectively. Let $P_{X|Y=y}$ denote the conditional probability density function of X given the value y of Y . We will sometimes write $D_\gamma(X)$ as shorthand for $D_\gamma(P_X)$, and we will define

$$D_\gamma(X | Y) = \mathbb{E}_y[D(P_{X|Y=y} \parallel \gamma)].$$

For a vector $x \in \mathbb{R}^n$ and a linear subspace $V \subseteq \mathbb{R}^n$, we denote the orthogonal projection of x onto V by $\text{proj}_V x$. We denote the Euclidean norm of x by $\|x\|$.

The Setup. For a contradiction, we begin by assuming the negation of Lemma 4.1. That is, we assume that there is a constant $b \geq 66$ such that for all constants $\delta > 0$, there exist $A, B \subseteq \{-1, 1\}^n$ such that

$$\min\{|A|, |B|\} \geq 2^{n-\delta n} \text{ and} \tag{17}$$

$$\Pr_{(X,Y) \in RA \times B} [\langle X, Y \rangle \notin [-b\sqrt{n}, b\sqrt{n}]] < \varepsilon := \text{tail}(2.01b). \tag{18}$$

We treat the sets A and B asymmetrically in the proof. Using the largeness of A , and appealing to a concentration inequality of Talagrand, we identify a subset $V \subseteq A$ consisting of $\Theta(n)$ vectors such that

(P1) the vectors in V are, in some sense, near-orthogonal; and

(P2) the quantity $\langle x, Y \rangle$, where $y \in_R B$, is concentrated around zero for *each* $x \in V$, in the sense of (18).

This step is a simple generalization of the first part of Sherstov's argument in his proof that $R(\text{GHD}_n) = \Omega(n)$.

As for the set B , we consider its *Gaussian analogue* $\tilde{B} := \{\tilde{y} \in \mathbb{R}^n : \text{sign}(\tilde{y}) \in B\}$. Consider the random variable $Q_x = \langle x, \tilde{Y} \rangle / \sqrt{n}$, for an arbitrary $x \in V$ and $\tilde{Y} \sim \mathcal{N}^n|_{\tilde{B}}$. On the one hand, we can show that property (P2) above implies “concentration” for Q_x in some sense. Combined with property (P1), we have that projections of the set \tilde{B} along $\Omega(n)$ near-orthogonal directions are all “concentrated.” On the other hand, arguing along the lines of Chakrabarti-Regev, we cannot have too much concentration along so many near-orthogonal directions, because \tilde{B} is a “large” subset of \mathbb{R}^n . The incompatibility of these two behaviors of Q_x gives us our desired contradiction.

It remains to identify a suitable notion of “concentration” that lets us carry out the above program. The notion we choose is the *escape probability* $p^* = \Pr[|Q_x| > (c + \alpha)b]$, for suitable constants $c, \alpha > 0$ that we shall determine later.

5.2 The Actual Proof

Let Y denote a uniformly distributed vector in B . Define the set

$$C := \{x \in A : \Pr_{Y \in_R B} [\langle x, Y \rangle \notin [-b\sqrt{n}, b\sqrt{n}]] < 2\varepsilon\}. \tag{19}$$

By Eq. (18) and Markov's inequality, we have $|C| \geq \frac{1}{2}|A| \geq 2^{n-\delta n-1}$. We now use some geometry.

Fact 5.1 (Generalization of [She11a, Lemma 3.1]). *Let $\delta > 0$ be a sufficiently small constant and let n be large enough. Put $k = \lceil \sqrt{\delta n} \rceil$. Suppose $C \subseteq \{-1, 1\}^n$ has size $|C| \geq 2^{n-\delta n-1}$. Then there exist $x_1, \dots, x_k \in A$ such that*

$$\forall j \in \{1, \dots, k\}, \text{ we have } \|\text{proj}_{\text{span}\{x_1, x_2, \dots, x_{j-1}\}} x_j\| \leq 2\delta^{1/4} \sqrt{n}, \tag{20}$$

Proof. Having chosen x_1, \dots, x_{j-1} (where $j \leq k$), we apply the appropriate variant of Talagrand's concentration inequality [AS00, Theorem 7.6.1] to obtain that $\|\text{proj}_{\text{span}\{x_1, \dots, x_{j-1}\}} x_j\|$ is sharply concentrated around $\sqrt{\dim \text{span}\{x_1, \dots, x_{j-1}\}} \leq \sqrt{k}$. In particular, there is an absolute constant c such that

$$\Pr_{x_j \in R\{-1,1\}^n} \left[\|\text{proj}_{\text{span}\{x_1, \dots, x_{j-1}\}} x_j\| > 2\delta^{1/4} \sqrt{n} \right] \leq 2^{-c\sqrt{\delta n}}.$$

On the other hand $\Pr_{x \in R\{-1,1\}^n} [x \in A] \geq 2^{-\delta n - 1}$, which is larger than the above estimate if δ is sufficiently small. Therefore, we can pick a suitable x_j to continue. \square

From now on, fix the “near-orthogonal” set of vectors x_1, \dots, x_k , with $k = \lceil \sqrt{\delta n} \rceil$, given by Fact 5.1. Recall that $\tilde{B} := \{\tilde{y} \in \mathbb{R}^n : \text{sign}(\tilde{y}) \in B\}$. We define a random variable \tilde{Y} correlated with Y as follows. Let (Y_1, \dots, Y_n) be the coordinates of Y ; then define $\tilde{Y}_j = Y_j |W_j|$, where $W_j \sim \gamma$ and put $\tilde{Y} = (\tilde{Y}_1, \dots, \tilde{Y}_n)$. Notice that the resulting distribution of \tilde{Y} is exactly $\gamma^n|_{\tilde{B}}$. We now define the random variable Q_j and its escape probability p_j^* as follows.

$$Q_j := \frac{\langle x_j, \tilde{Y} \rangle}{\sqrt{n}}; \quad p_j^* := \Pr[|Q_j| > (c + \alpha)b].$$

We shall eventually fix a particular index j and choose suitable constants c and α above. As mentioned in the overview, the proof will hinge on a careful analysis of this escape probability.

Lower Bounding the Escape Probability

We begin the study by showing that there exists an index $j \in \{1, \dots, k\}$ such that Q_j behaves quite similarly to a mixture of shifted standard normal variables (i.e., variances close to 1, but arbitrary means). This will in turn yield a lower bound on the corresponding p_j^* .

Let $\tilde{x}_1, \dots, \tilde{x}_k$ be the (truly) orthogonal vectors obtained from x_1, \dots, x_k by the Gram-Schmidt process, i.e., $\tilde{x}_i := x_i - \text{proj}_{\text{span}\{x_1, \dots, x_{i-1}\}} x_i$. For $i \in \{1, \dots, k\}$, put $x_i^* = \tilde{x}_i / \|\tilde{x}_i\|$, and let x_{k+1}^*, \dots, x_n^* be a completion of these vectors to an *orthonormal* basis of \mathbb{R}^n . Expressing \tilde{Y} in this basis, and noting that $\langle x_j, x_i^* \rangle = 0$ for all $i > j$ in step (21) below, we derive

$$\begin{aligned} Q_j &= \frac{1}{\sqrt{n}} \left\langle x_j, \sum_{i=1}^n \langle \tilde{Y}, x_i^* \rangle x_i^* \right\rangle \\ &= \sum_{i=1}^j \frac{\langle x_j, x_i^* \rangle}{\sqrt{n}} \langle \tilde{Y}, x_i^* \rangle \end{aligned} \tag{21}$$

$$\begin{aligned} &= \frac{\langle x_j, x_j^* \rangle}{\sqrt{n}} \langle \tilde{Y}, x_j^* \rangle + \sum_{i=1}^{j-1} \frac{\langle x_j, x_i^* \rangle}{\sqrt{n}} \langle \tilde{Y}, x_i^* \rangle \\ &= r_j Z_j + S_j, \end{aligned} \tag{22}$$

where we define

$$r_j := \frac{\langle x_j, x_j^* \rangle}{\sqrt{n}}, \quad Z_j := \langle \tilde{Y}, x_j^* \rangle, \quad S_j := \sum_{i=1}^{j-1} \frac{\langle x_j, x_i^* \rangle}{\sqrt{n}} \langle \tilde{Y}, x_i^* \rangle.$$

The Pythagorean theorem says that $\langle x_j, x_j^* \rangle^2 = \|x_j\|^2 - \|\text{proj}_{\text{span}\{x_1, x_2, \dots, x_{j-1}\}} x_j\|^2$. Recalling that $\|x_j\| = \sqrt{n}$ and using (20), we conclude that

$$\forall j \in \{1, \dots, k\}, \text{ we have } 1 - 4\sqrt{\delta} \leq r_j \leq 1. \tag{23}$$

Lemma 5.2. *There exists $j \in \{1, \dots, k\}$ such that $D_\gamma(Z_j | S_j) \leq \sqrt{\delta}$.*

Proof. Since $|B| \geq 2^{n-\delta n}$, we have $\gamma^n(\tilde{B}) \geq 2^{-\delta n}$. By definition, we have $D_\gamma(\gamma^n|_{\tilde{B}}) = -\ln \gamma^n(\tilde{B}) \leq (\ln 2)\delta n \leq \delta n$. On the other hand, by the chain rule for relative entropy, we have

$$D_\gamma(\gamma^n|_{\tilde{B}}) = D_\gamma(\tilde{Y}) = D_\gamma\left(\langle \tilde{Y}, x_1^* \rangle, \dots, \langle \tilde{Y}, x_n^* \rangle\right) = \sum_{j=1}^n D_\gamma\left(\langle \tilde{Y}, x_j^* \rangle \mid \langle \tilde{Y}, x_1^* \rangle, \dots, \langle \tilde{Y}, x_{j-1}^* \rangle\right).$$

Recalling that $k = \lceil \sqrt{\delta n} \rceil$, we deduce that there exists an index $j \in \{1, \dots, k\}$ such that

$$D_\gamma\left(\langle \tilde{Y}, x_j^* \rangle \mid \langle \tilde{Y}, x_1^* \rangle, \dots, \langle \tilde{Y}, x_{j-1}^* \rangle\right) \leq \sqrt{\delta}. \quad (24)$$

Since S_j is a function of $\langle \tilde{Y}, x_1^* \rangle, \dots, \langle \tilde{Y}, x_{j-1}^* \rangle$, we conclude that $D_\gamma(Z_j \mid S_j) \leq \sqrt{\delta}$. \square

For the rest of our proof, we fix an index j as guaranteed by Lemma 5.2. We put $r = r_j, Z = Z_j, S = S_j, Q = Q_j$, and $p^* = p_j^*$. Now define the set

$$\mathcal{S} = \{s \in \mathbb{R} : D_\gamma(Z \mid S = s) \leq \delta^{1/4}\},$$

so that $\Pr[S \notin \mathcal{S}] \leq \delta^{1/4}$ by Markov's inequality. Clearly, either $\Pr[S \geq 0 \mid S \in \mathcal{S}] \geq \frac{1}{2}$ or $\Pr[S \leq 0 \mid S \in \mathcal{S}] \geq \frac{1}{2}$. In what follows, we shall assume that the former condition holds; it will soon be clear that this does not lose generality. Under this assumption we have

$$D_\gamma(Z \mid S \geq 0 \wedge S \in \mathcal{S}) \leq 2\delta^{1/4}. \quad (25)$$

Therefore, by Pinsker's inequality [CT06], the statistical distance between the distribution γ and the distribution of $(Z \mid S \geq 0 \wedge S \in \mathcal{S})$ is at most $\sqrt{2(2\delta^{1/4})} = 2\delta^{1/8}$. Using this fact below, we get

$$\begin{aligned} p^* &\geq \Pr[Q > (c + \alpha)b] \\ &= \Pr[rZ + S \geq (c + \alpha)b \mid S \geq 0 \wedge S \in \mathcal{S}] \cdot \Pr[S \geq 0 \mid S \in \mathcal{S}] \cdot \Pr[S \in \mathcal{S}] \\ &\geq \frac{1}{2}(1 - \delta^{1/4}) \Pr[rZ + S \geq (c + \alpha)b \mid S \geq 0 \wedge S \in \mathcal{S}] \\ &\geq \frac{1}{2}(1 - \delta^{1/4}) \Pr[Z \geq (c + \alpha)b/r \mid S \geq 0 \wedge S \in \mathcal{S}] \\ &\geq \frac{1}{2}(1 - \delta^{1/4}) (\text{tail}((c + \alpha)b/r) - 2\delta^{1/8}) \\ &\geq \frac{1 - \delta^{1/4}}{2} \left(\text{tail}\left(\frac{(c + \alpha)b}{1 - 4\sqrt{\delta}}\right) - 2\delta^{1/8} \right), \end{aligned} \quad (26)$$

where the final step uses the lower bound on r given by (23).

Upper Bounding the Escape Probability

Recall that we had fixed a specific index j after the proof of Lemma 5.2, and that $Q = Q_j = \langle x_j, \tilde{Y} \rangle / \sqrt{n}$. We shall now explore the relation between $\langle x_j, Y \rangle$ and $\langle x_j, \tilde{Y} \rangle$ to upper bound the escape probability. At this point it would help to review the discussion of the relation between Y and \tilde{Y} at the beginning of Section 5.2.

For simplicity, we put $x := x_j$ and assume, w.l.o.g., that $x = (1, 1, \dots, 1)$ so that $\langle x, y \rangle = \sum_{i=1}^n y_i$. This is legitimate because, if $x_i = -1$, we can flip x_i to 1 and y_i to $-y_i$ without changing $\langle x, y \rangle$.

Recall that each coordinate \tilde{Y}_i of \tilde{Y} has the same distribution as $Y_i|W_i|$, where the variables $\{W_i\}$ are independent and each $W_i \sim \gamma$. Define $T := \sum_{i=1}^n Y_i / \sqrt{n}$; note that T is a *discrete* random variable. After some reordering of coordinates, we can rewrite

$$\sqrt{n}Q = \langle x, \tilde{Y} \rangle = \left(|W_1| + |W_2| + \dots + |W_{\frac{n}{2} + \frac{T\sqrt{n}}{2}}| \right) - \left(|W_{\frac{n}{2} + \frac{T\sqrt{n}}{2} + 1}| + \dots + |W_n| \right).$$

Each $|W_i|$ has a so-called *half normal* distribution. This is a well-studied distribution: in particular, for each i , we know that

$$\mathbb{E}[|W_i|] = \sqrt{\frac{2}{\pi}}, \quad \text{Var}[|W_i|] = 1 - \frac{2}{\pi}.$$

Thus, for each value t in the range of T , we have $\mathbb{E}[\sqrt{n}Q \mid T = t] = t\sqrt{2n/\pi}$ by linearity of expectation, and $\text{Var}[\sqrt{n}Q \mid T = t] = (1 - 2/\pi)n$ by the independence of the variables $\{W_i\}$. The half-normal distribution is well-behaved enough for us to apply Lindeberg's version of the central limit theorem [Fel68]: doing so tells us that as n grows, the distribution of

$$\frac{\sqrt{n}Q^{(t)} - \mathbb{E}[\sqrt{n}Q^{(t)}]}{\sqrt{\text{Var}[\sqrt{n}Q^{(t)}]}} = \frac{Q^{(t)} - t\sqrt{2/\pi}}{\sqrt{1 - 2/\pi}}$$

converges to γ , where $Q^{(t)} = (Q \mid T = t)$. In other words, the distribution of $Q^{(t)}$ converges to the (shifted and scaled) normal distribution $\mathcal{N}(t\sqrt{2/\pi}, 1 - 2/\pi)$. Therefore, the distribution of Q converges to a mixture of such distributions. Fix the constants

$$c := \sqrt{2/\pi}; \quad \sigma := \sqrt{1 - 2/\pi}.$$

Then the distribution of Q converges to that of $V + cT$, where $V \sim \mathcal{N}(0, \sigma^2)$ is independent of T . Using the convergence, we can easily prove the following claim.

Claim 5.3. *For sufficiently large n , we have $p^* = \Pr[|Q| > (c + \alpha)b] \leq 2\Pr[|V + cT| > (c + \alpha)b]$.* \square

Recalling that $x \in C$, and using (19), we have $\Pr[|T| > b] \leq 2\varepsilon$. This lets us upper bound p^* as follows.

$$\begin{aligned} \frac{p^*}{2} &\leq \Pr[|V + cT| > (c + \alpha)b \mid |T| \leq b] + \Pr[|T| > b] \\ &\leq \Pr[|V| > \alpha b] + \Pr[|T| > b] \\ &\leq 2\Pr[V/\sigma > (\alpha/\sigma)b] + 2\varepsilon \\ &= 2\text{tail}((\alpha/\sigma)b) + 2\text{tail}(2.01b), \end{aligned} \tag{27}$$

where in the last step we use the definition of ε as given in (18).

Completing the Proof

To complete the proof of the anti-concentration lemma, we combine the lower bound (26) with the upper bound (27) to obtain

$$\frac{1 - \delta^{1/4}}{2} \left(\text{tail}\left(\frac{(c + \alpha)b}{1 - 4\sqrt{\delta}}\right) - 2\delta^{1/8} \right) \leq 4\text{tail}\left(\frac{\alpha b}{\sigma}\right) + 4\text{tail}(2.01b).$$

Recall that we had started by assuming the negation of Lemma 4.1, in Eqs. (17) and (18). Thus, the above inequality is supposed to hold for some constant $b \geq 66$ and all constants $\delta > 0$. However, if set $\alpha = 2.01\sigma$, we can get a contradiction: as $\delta \rightarrow 0$, the left-hand side approaches $\frac{1}{2}\text{tail}((c + 2.01\sigma)b)$, whereas the right-hand side is $8\text{tail}(2.01b)$. Plugging in the values of c and σ , we note that $c + 2.01\sigma < 2.01$. Therefore, if we choose δ small enough, we have a contradiction.

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